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Reducibility of a class of nonlinear kinetic integral equations

Damián H Zanette

Grupo de Física Estadística, División Física Teórica, Centro Atómico Bariloche and Instituto Balseiro, 8400 Bariloche, Argentina

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Abstract. The reducibility to differential forms of certain nonlinear integral operators appearing in kinetic theory is discussed in the frame of a stationary problem in gas dynamics with removal events. A wide class of transition kernels allowing such reducibility is characterized, and the validity of these kernels as approximations to real ones is studied in terms of their asymptotic behaviour for large and small velocities. In fact, this behaviour is shown to determine the corresponding limiting velocity dependence in the solutions to the kinetic equation.

1. Introduction

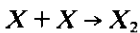
The bilinear integral operator

$$\mathcal{K}[f] = f(v) \int_0^\infty K(v, w) f(w) dw \quad (1.1)$$

acting on a function $f(v)$ ($0 \leq v < \infty$), with a given kernel $K(v, w)$ which satisfies

$$K(v, w) = K(w, v) \quad (1.1a)$$

appears, as well as similar integral forms, in a variety of problems related with kinetic theory [1, 2]. Consider, for instance, the following situation arising in extended kinetic theory [3, 4]: The particles of a spatially homogeneous, velocity isotropic gas undergo removal collisions, i.e. binary interactions, in which the particles are destroyed. This is the case, for example, for an atomic gaseous species X subject to the reaction



in which, from the point of view of the atomic distribution, the formation of molecules X_2 can be seen as 'destruction' of atoms. The same situation arises when considering ionic recombination. Of course, if this removal is the only process acting on the gas, the system evolves towards a vanishing distribution state. This continuous depletion in the gas distribution can, however, be compensated with the addition of an external source, providing particles to the system with a fixed distribution. The competition between removal and source determines a non-trivial evolution, characterized in fact by a stationary state which satisfies the nonlinear integral equation

$$\mathcal{K}[f] = S(v) \quad (1.2)$$

where $f(v)$ is now the (time-independent) distribution function for the gas, depending on the velocity modulus v , and $S(v)$ is the source distribution. In this problem, the

kernel $K(v, w)$ of (1.1) represents the transition frequency of a removal event by collision of two particles with velocities v and w , respectively. The symmetry (1.1a) accounts now for the identity of the involved particles. As in most physical applications, both $S(v)$ and $K(v, w)$, as well as the solution $f(v)$, are required to be non-negative functions of their arguments.

The rigorous expression for the kernel $K(v, w)$ is obtained from the microscopic transition probability $I(|v-w|) = |v-w|\sigma(|v-w|)$, depending on the modulus of the relative incoming velocity $u = |v-w|$ through the collision cross section $\sigma(u)$, whose angular dependence has been eliminated by integration. Indeed, the transition kernel is given by [5]

$$K(v, w) = \frac{1}{2vw} \int_{|v-w|}^{v+w} uI(u) du \quad (1.3)$$

as an integral between the extreme values attained by the relative incoming velocity. Equation (1.3) determines, in general, a very awkward structure for $K(v, w)$. In fact, except for some very special forms of $I(u)$ (see section 2), this function is typically expressed in two parts, according to how v and w compare, as implied by the absolute value in the lower limit of the integral. This fact is usually neglected when proposing model expressions for the transition kernel [5], in the hope that this will not affect considerably the main characteristics of the solutions to the kinetic equations. However, it is worthwhile to remark that such two-folded structure accounts for the fact that for fixed v and w , $K(v, w)$ is only determined by the values of $I(u)$ in a bounded interval of the variable u , so that a model expression for the transition kernel which does not present such structure cannot be expected to derive from a microscopic cross-section. In any case, even for the simplest models, the solution of the related integral equations implies the use of proper numerical techniques, as an analytical treatment is impracticable.

In this paper, the problem of dealing with an equation like (1.2) is attacked from a different point of view. In fact, a systematic procedure is developed for reducing the integral operator to a nonlinear differential form for a new dependent variable. Accordingly, suitable boundary conditions are derived, in order to completely determine the differential problem. The scheme is applicable to a certain class of symmetric transition kernels, which are chosen to present the two-folded structure implied by (1.3) and characterized in the main part of the paper. It is clear that the reduction of the kinetic equation to a differential form enables the application of a wide toolkit of standard analytical and numerical analysis. Therefore, particular attention is paid to the validity of such kernels as models for real processes. The extension to other kinetic equations of interest is also discussed.

2. Previous results on equation (1.2)

Before passing to the exposition of the reduction method from integral to differential form for the nonlinear operator (1.1), it is convenient to include a brief review on known solutions for equation (1.2), for arbitrary forms of either the source term $S(v)$ or the kernel $K(v, w)$. These solutions will be used to compare with the results obtained later.

Even when existence and uniqueness of solutions to (1.2) are easily proven to hold, the question of finding its explicit solution is indeed a very hard one. From the physical

point of view, the most interesting question in dealing with this stationary kinetic equation is to obtain the distribution $f(v)$ as a function of $S(v)$ for a given symmetric transition kernel. Unfortunately, the forms of $K(v, w)$ allowing for such a general solution are scarce, and do not seem to be particularly representative of real problems. Two special classes deserve to be quoted:

(i) $K(v, w)$ separable with respect to multiplication, i.e.

$$K(v, w) = k(v)k(w) \tag{2.1}$$

for which the solution to (1.2) is given by

$$f(v) = \frac{S(v)}{k(v)\langle k \rangle} \tag{2.2a}$$

with

$$\langle k \rangle = \int_0^\infty k(v)f(v) dv \tag{2.2b}$$

the mean value of $k(v)$ over the distribution function. This quantity has to be determined consistently from equations (2.2), and reads

$$\langle k \rangle^2 = \int_0^\infty S(v) dv. \tag{2.2c}$$

A particularly trivial situation in which $K(v, w)$ has the form (2.1) is obtained for $I(u) = 1$ (cf (1.3)), which defines the Maxwell interaction model [6], and implies $K(v, w) = 1$. In this case, $\langle k \rangle = \langle 1 \rangle$ represents the number density of the gas.

(ii) $K(v, w)$ separable with respect to sum, i.e.

$$K(v, w) = k(v) + k(w) \tag{2.3}$$

determining

$$f(v) = \frac{S(v)}{\langle 1 \rangle k(v) + \langle k \rangle}. \tag{2.4a}$$

The density $\langle 1 \rangle$ and the mean value $\langle k \rangle$ are the solutions to the following two functional equations:

$$\langle 1 \rangle = \int_0^\infty \frac{S(v)}{\langle 1 \rangle k(v) + \langle k \rangle} dv \tag{2.4b}$$

$$\langle k \rangle = \int_0^\infty \frac{k(v)S(v)}{\langle 1 \rangle k(v) + \langle k \rangle} dv.$$

Within this class of separable kernels one finds the ‘very-hard-particle’ model [6], $I(u) = u^2$, which produces $K(v, w) = v^2 + w^2$.

These two cases of solvable models can be put together in the following symmetric form:

$$K(v, w) = \sum_{i,j=1}^N a_{ij}k_i(v)k_j(w) \tag{2.5}$$

where $a_{ij} = a_{ji}$, N is arbitrary, and the functions $k_i(v)$ are chosen so that $K(v, w)$ is positive. In this generalization, the solution to (1.2) reads

$$f(v) = \frac{S(v)}{\sum_{ij} a_{ij} \langle k_j \rangle k_i(v)} \quad (2.6a)$$

and the constants $\langle k_n \rangle$ are obtained from the set of functional equations

$$\langle k_n \rangle = \int_0^\infty \frac{k_n(v) S(v)}{\sum_{ij} a_{ij} \langle k_j \rangle k_i(v)} dv \quad (n = 1, 2, \dots, N). \quad (2.6b)$$

From a mathematical viewpoint, it is also interesting to find solutions to (1.2) as a function of the kernel $K(v, w)$, for a given form of the source $S(v)$. To the author's knowledge, however, there is only one known form of $S(v)$ allowing for a general solution to the stationary kinetic equation, namely

$$S(v) = S_0 \delta(v - v_0) \quad (2.7)$$

i.e. a monochromatic source. Provided that $K(v_0, v_0) \neq 0$, the solution in this situation is

$$f(v) = \sqrt{\frac{S_0}{K(v_0, v_0)}} \delta(v - v_0) \quad (2.8)$$

so that, as expected, in the stationary state all particles have velocity v_0 . Since this solution holds for arbitrary forms of $K(v, w)$, it is an ideal tool for checking the results obtained in the following.

3. Reduction to the differential form

Consider the class of symmetric kernels with two-folded structure:

$$K(v, w) = \begin{cases} \psi(v, w) k(v) k(w) & \text{for } w < v \\ \psi(w, v) k(v) k(w) & \text{for } w > v \end{cases} \quad (3.1)$$

where $\psi(v, w)$ and $k(v)$ are chosen so that $K(v, w)$ is positive. The stationary kinetic equation (1.2) reads, in this situation

$$f(v) k(v) \left[\int_0^v \psi(v, w) k(w) f(w) dw + \int_v^\infty \psi(w, v) k(w) f(w) dw \right] = S(v). \quad (3.2)$$

In order to reduce this equation to a differential form, the dependent variable is changed to

$$g(v) = \int_0^v \psi(v, w) k(w) f(w) dw + \int_v^\infty \psi(w, v) k(w) f(w) dw \quad (3.3)$$

which is just the factor in the square brackets of (3.2). The idea is now to obtain the function $f(v)$ from a derivative of $g(v)$, whose order must be properly chosen. The first derivative of the new unknown $g(v)$, as expressed by (3.3), is given by

$$\partial_v g(v) = \int_0^v \partial_1 \psi(v, w) k(w) f(w) dw + \int_v^\infty \partial_2 \psi(w, v) k(w) f(w) dw \quad (3.4)$$

where ∂_1 and ∂_2 indicate derivation with respect to the first and the second variable,

respectively. Accordingly, the $(n + 1)$ th order derivative of $g(v)$ reads

$$\partial_v^{n+1}g(v) = \sum_{m=0}^n \partial_v^m \{ [\partial_1^{n-m}\psi(v, v) - \partial_2^{n-m}\psi(v, v)]k(v)f(v) \} + \int_0^v \partial_1^{n+1}\psi(v, w)k(w)f(w) dw + \int_v^\infty \partial_2^{n+1}\psi(w, v)k(w)f(w) dw. \quad (3.5)$$

This derivative can be seen to be proportional to $k(v)f(v)$ by requiring the following conditions:

$$\partial_1^{n+1}\psi(v, w) = \partial_2^{n+1}\psi(v, w) = 0 \quad (3.6a)$$

and

$$\partial_1^m\psi(v, v) - \partial_2^m\psi(v, v) = 0 \quad (3.6b)$$

for $m = 1, 2, \dots, n - 1$. Indeed, if equations (3.6) are fulfilled, one obtains

$$\partial_v^{n+1}g(v) = [\partial_1^n\psi(v, v) - \partial_2^n\psi(v, v)]k(v)f(v) \quad (3.7)$$

as in the RHS of (3.5) the only contribution comes from the term with $m = 0$ in the summation. In this situation, the integral equation (3.2) reduces to

$$g(v)\partial_v^{n+1}g(v) = [\partial_1^n\psi(v, v) - \partial_2^n\psi(v, v)]S(v) \quad (3.8)$$

which is a nonlinear ordinary differential equation for $g(v)$. Observe that, by virtue of (3.6a), the square bracket in the RHS is a constant, as its derivative is identically zero.

Of course, the function $\psi(v, w)$ has to have a very particular form in order to fulfill conditions (3.6). In fact, the characterization of the functions satisfying those conditions is the next problem to be dealt with. One observes then that (3.6a), which requires the $(n + 1)$ th derivatives of ψ to vanish, implies that this function is a polynomial whose degree is at most equal to n . For simplicity, it is taken to be a homogeneous polynomial of n th degree, i.e.

$$\psi(v, w) = \sum_{r=0}^n \psi_r v^r w^{n-r}. \quad (3.9)$$

The generalizations discussed in section 5 relax this particular choice.

Differentiating conveniently with respect to v and w yields the form of conditions (3.6b), required for $\psi(v, w)$ as given by (3.9), namely,

$$\sum_{r=0}^n r^m (\psi_r - \psi_{n-r}) = 0 \quad (3.10)$$

for $m = 1, 2, \dots, n - 1$. A relevant question regarding the freedom in the choice of a suitable form for $\psi(v, w)$ is whether conditions (3.10) are independent or not. It can be seen that, for m even, the corresponding condition (3.10) can be derived from those obtained for $q < m$. In fact, one obtains successively

$$\begin{aligned} \sum_{r=0}^n r^m (\psi_r - \psi_{n-r}) &= \sum_{r=0}^n (n-r)^m (\psi_{n-r} - \psi_r) \\ &= \sum_{q=0}^m \binom{m}{q} (-1)^q n^{m-q} \sum_{r=0}^n r^q (\psi_{n-r} - \psi_r) \\ &= n^m \sum_{r=0}^n (\psi_{n-r} - \psi_r) + (-1)^m \sum_{r=0}^n r^m (\psi_{n-r} - \psi_r) \\ &= (-1)^{m+1} \sum_{r=0}^n r^m (\psi_r - \psi_{n-r}). \end{aligned} \quad (3.11)$$

Comparing the extremes of the sequence (3.11) it is clear that, for m even, (3.10) is satisfied automatically. This fact implies that, among the $n - 1$ conditions (3.10), only $[n/2]$ of them are independent ($[x]$ indicates the integer part of x). There is, however, a more relevant implication deriving from this dependence: if n is even, the accomplishment of (3.10) implies that $\sum_r r^n (\psi_r - \psi_{n-r}) = 0$. But this is exactly the factor multiplying $S(v)$ in (3.8), so that the $(n + 1)$ th derivative of $g(v)$ vanishes, and its identification with $f(v)$ is not possible any more. Consequently, one must be restricted to work with n odd. This fact is a direct consequence of the symmetry (1.1a), imposed on the transition kernel. Indeed, if this condition is relaxed, n could be an even integer.

For n odd, (3.10) imply then $(n - 1)/2$ conditions to be satisfied by the $(n + 1)/2 = 1 + (n - 1)/2$ differences $\psi_r - \psi_{n-r}$. Therefore, all the solutions to (3.10) are multiples of a fundamental one. From a well known property of combinatorial numbers [7], a particular solution is

$$\psi_r - \psi_{n-r} = (-1)^{n-r} \binom{n}{r} \tag{3.12a}$$

so that the general solution can be written as

$$\psi_r - \psi_{n-r} = (-1)^{n-r} (\psi_n - \psi_0) \binom{n}{r} \tag{3.12b}$$

which determines $(n - 1)/2$ conditions to be satisfied by the $n + 1$ coefficients ψ_r . Then, $(n + 3)/2$ of them can be freely chosen. This solves the problem of determining the possible forms of the function $\psi(v, w)$ allowing the identification (3.7) of $\partial_v^{n+1} g(v)$ with $f(v)$, and therefore reducing the integral equation (3.2) to its differential form. Taking into account equations (3.8) and (3.9), it reads

$$g(v) \partial_v^{n+1} g(v) = n! (\psi_n - \psi_0) S(v). \tag{3.13}$$

There remains now one last question to be dealt with, namely, whether (3.13) is completely equivalent to the integral form (3.2). In fact, a particular solution to (3.13) is well determined only after $n + 1$ boundary conditions have been imposed. These boundary conditions imply a wide freedom in the choice of the solution to (3.13), which could be in contradiction to the fact that $g(v)$ must be associated to the integral operator in (3.2) [cf (3.3)]. Therefore, one must determine the proper boundary conditions to be used in solving (3.13), according to the form imposed on the solution $g(v)$ by (3.3). This problem is treated as follows: Consider the function

$$g(v) = (\psi_n - \psi_0) \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} v^r \int_0^v w^{n-r} k(w) f(w) dw + \sum_{r=0}^n \partial_v^r g(0) \frac{v^r}{r!}. \tag{3.14}$$

This is the most general function satisfying (3.7). In fact, independently of the boundary values $\partial_v^r g(0)$ ($r = 0, 1, \dots, n$), its $(n + 1)$ th derivative is proportional to $k(v)f(v)$ with the proper factor. Now, this value must be chosen so that $g(v)$ satisfy (3.3), which, according to (3.9), takes the form

$$g(v) = \sum_{r=0}^n \psi_r v^r \int_0^v w^{n-r} k(w) f(w) dw + \sum_{r=0}^n \psi_{n-r} v^r \int_v^\infty w^{n-r} k(w) f(w) dw. \tag{3.15}$$

Subtracting (3.15) from (3.14) and making use of the solutions (3.12b) one gets

successively

$$\begin{aligned}
 0 &= \sum_{r=0}^n v^r \left[(-1)^{n-r} (\psi_n - \psi_0) \binom{n}{r} \int_0^v w^{n-r} k(w) f(w) w^{n-r} dw \right. \\
 &\quad \left. - \psi_r \int_0^v w^{n-r} k(w) f(w) dw - \psi_{n-r} \int_v^\infty w^{n-r} k(w) f(w) dw \right] + \sum_{r=0}^n \partial_v^r g(0) \frac{v^r}{r!} \\
 &= - \sum_{r=0}^n v^r \psi_{n-r} \int_0^\infty w^{n-r} k(w) f(w) dw + \sum_{r=0}^n \partial_v^r g(0) \frac{v^r}{r!} \\
 &= - \sum_{r=0}^n v^r \psi_{n-r} \langle v^{n-r} k \rangle + \sum_{r=0}^n \partial_v^r g(0) \frac{v^r}{r!} \tag{3.16}
 \end{aligned}$$

where the mean values $\langle v^r k \rangle$ are defined according to (2.2b). The proper choice for the boundary values of $g(v)$ and its derivatives is then

$$\partial_v^r g(0) = r! \psi_{n-r} \langle v^{n-r} k \rangle \quad (r=0, 1, \dots, n) \tag{3.17}$$

completing the equivalence between the differential problem (3.13) and the original integral one.

4. Summary and examples

It is convenient to resume the steps of the reduction of the integral equation to its differential form in the following manner: One starts with a nonlinear integral equation of the form

$$f(v) \int_0^\infty K(v, w) f(w) dw = S(v) \tag{4.1}$$

for the unknown $f(v)$, with given positive source $S(v)$ and kernel $K(v, w)$. This latter function is given by

$$K(v, w) = \begin{cases} \psi(v, w) k(v) k(w) & \text{for } w < v \\ \psi(w, v) k(v) k(w) & \text{for } w > v \end{cases} \tag{4.2}$$

where $\psi(v, w)$ is an n th degree homogeneous polynomial (n odd) and

$$\psi(v, w) = \sum_{r=0}^n \psi_r v^r w^{n-r} \tag{4.2a}$$

whose coefficients satisfy

$$\psi_r - \psi_{n-r} = (-1)^{n-r} (\psi_n - \psi_0) \binom{n}{r}. \tag{4.2b}$$

Under these conditions, the integral equation can be reduced to the nonlinear $(n + 1)$ th degree differential form

$$g(v) \partial_v^{n+1} g(v) = n! (\psi_n - \psi_0) S(v) \tag{4.3}$$

for a new unknown $g(v)$, which just coincides with the integral factor in the original equation, and is related to $f(v)$ through

$$\partial_v^{n+1} g(v) = n! (\psi_n - \psi_0) k(v) f(v). \tag{4.4}$$

Finally, the differential equation obtained has to be solved with the boundary conditions

$$\partial_v^r g(0) = r! \psi_{n-r} \langle v^{n-r} k \rangle \tag{4.5}$$

for $r = 0, 1, \dots, n$. Note that, as in the examples developed in section 2, the mean values $\langle v^r k \rangle$ are not known *a priori*, and must be determined consistently once $f(v)$ has been found.

In order to illustrate the procedure proposed above, the case with $n = 3$ is developed explicitly hereafter. The possible forms of the transition kernel leading to the fourth-order nonlinear equation obtained in (3.13) contain then a third-degree polynomial reading

$$\psi(v, w) = \psi_0 w^3 + \psi_1 v w^2 + \psi_2 v^2 w + \psi_3 v^3. \tag{4.6}$$

According to (3.10), the coefficients ψ_r must satisfy the following conditions:

$$(\psi_1 - \psi_2) + 2(\psi_2 - \psi_1) + 3(\psi_3 - \psi_0) = 0 \tag{4.7a}$$

$$(\psi_1 - \psi_2) + 4(\psi_2 - \psi_1) + 9(\psi_3 - \psi_0) = 0 \tag{4.7b}$$

which, certainly, are not independent, as discussed before. Imposing these conditions on $\psi(v, w)$ yields

$$\psi(v, w) = \psi_0 w^3 + \psi_1 v w^2 + \psi_2 v^2 w + [\psi_0 + \frac{1}{3}(\psi_1 - \psi_2)] v^3 \tag{4.8}$$

so that three constants remain free to be fixed when determining the form of $K(v, w)$. Supposing that the function $k(v)$ in (3.1) is taken to be positive, the positivity of the kernel is ensured requiring that

$$\psi_0 + \psi_1 x + \psi_2 x^2 + [\psi_0 + \frac{1}{3}(\psi_1 - \psi_2)] x^3 \geq 0 \tag{4.9}$$

for all $x \geq 0$, which is accomplished if $\psi_0 \geq 0$, $\psi_2 - \psi_1 \leq 3\psi_0$ and if the polynomial in (4.9) has no real roots for $x > 0$.

Once these conditions are fulfilled, the integral equation can be reduced to the form

$$g(v) \partial_v^4 g(v) = 2(\psi_1 - \psi_2) S(v) \tag{4.10a}$$

to be solved with the following boundary conditions:

$$\begin{aligned} g(0) &= [\psi_0 + \frac{1}{3}(\psi_1 - \psi_2)] \langle v^3 k \rangle & \partial_v g(0) &= \psi_2 \langle v^2 k \rangle \\ \partial_v^2 g(0) &= 2\psi_1 \langle v k \rangle & \partial_v^3 g(0) &= 6\psi_0 \langle k \rangle. \end{aligned} \tag{4.10b}$$

The distribution function, solution to the integral equation, is then determined by

$$f(v) = \frac{1}{2k(v)(\psi_1 - \psi_2)} \partial_v^4 g(v). \tag{4.11}$$

From the point of view of the applications, a particularly important example belonging to the class of kernels with $n = 3$ is the well known *exact* ‘hard-sphere’ model [8]. In this model, the transition kernel is obtained from (1.3) with $I(u) = u$, and reads

$$K(v, w) = \begin{cases} v[1 + \frac{1}{3}(w/v)^2] & \text{for } w < v \\ w[1 + \frac{1}{3}(v/w)^2] & \text{for } w > v. \end{cases} \tag{4.12}$$

In the frame of the present formulation, the ‘hard-sphere’ model follows by putting $k(v) = v^{-1}$, and taking $\psi_0 = \frac{1}{3}$, $\psi_1 = 0$ and $\psi_2 = 1$, which implies $\psi_3 = 0$.

As expected from the discussion at the end of section 2, a general solution to (4.3) can be obtained for the case of a monochromatic source, i.e. for $S(v)$ as given in (2.7). In this case, in fact, the singular contribution in the LHS of (4.3) must come from $\partial_v^{n+1}g$, which is proposed to have the form

$$\partial_v^{n+1}g(v) = \gamma\delta(v - v_0) \tag{4.13}$$

where the constant γ must be determined consistently. Taking into account the boundary conditions (4.5), the function $g(v)$ reads

$$g(v) = \begin{cases} \sum_{r=0}^n \psi_{n-r} \langle v^{n-r}k \rangle v^r & \text{for } v < v_0 \\ \sum_{r=0}^n \psi_{n-r} \langle v^{n-r}k \rangle v^r + \gamma[(v - v_0)^n]/n! & \text{for } v > v_0. \end{cases} \tag{4.14}$$

Equation (4.4) enables the calculation of $f(v)$, which is also monochromatic, and fixes the mean values in the summations of (4.14) to be

$$\langle v^{n-r}k \rangle = \frac{\gamma v_0^{n-r}}{n!(\psi_n - \psi_0)}. \tag{4.14a}$$

The identity of the coefficients multiplying the delta functions in both sides of (4.3) fixes finally the value of γ according to

$$\left[\frac{\gamma}{n!(\psi_n - \psi_0)} \right]^2 v_0^n \sum_{r=0}^n \psi_{n-r} = S_0. \tag{4.15}$$

The distribution function is then

$$f(v) = \sqrt{\frac{S_0}{k^2(v_0)v_0^n \sum_{r=0}^n \psi_r}} \delta(v - v_0) \equiv \sqrt{\frac{S_0}{K(v_0, v_0)}} \delta(v - v_0) \tag{4.16}$$

in full agreement with (2.8).

5. Generalizations

By exploiting the linearity of the operator $\mathcal{K}[f]$ in (1.1) with respect to the transition kernel $K(v, w)$, two generalizations of the form of this kernel are possible. They are analysed hereafter.

5.1. Generalization I

Consider equations (3.14) to (3.16). One can see that if a term of the type $\phi(v, w)k(v)k(w)$, is added to the kernel proposed in (3.1) and (3.9), its effects can be absorbed by the initial conditions $\partial_v^r g(0)$ ($r = 0, 1, \dots, n$), if the function $\phi(v, w)$ is a symmetric polynomial of degree lower than or equal to n . Indeed, if

$$\tilde{K}(v, w) = K(v, w) + \phi(v, w)k(v)k(w) \tag{5.1}$$

and

$$\phi(v, w) = \sum_{s=0}^n \sum_{r=0}^s \phi_{sr} v^r w^{s-r} \tag{5.1a}$$

with $\phi_{sr} = \phi_{s, s-r}$, the corresponding new dependent variable

$$\tilde{g}(v) = k(v)^{-1} \int_0^\infty \tilde{K}(v, w) f(w) dw \tag{5.2}$$

can be also taken to satisfy (3.7), as the new terms do not affect its $(n + 1)$ th derivative.

Equating (3.14) and (3.15) for $\tilde{g}(v)$ yields

$$\sum_{r=0}^n v^r \psi_{n-r} \langle v^{n-r} k \rangle + \sum_{r=0}^n v^r \sum_{s=r}^n \phi_{sr} \langle v^{s-r} k \rangle = \sum_{r=0}^n \partial_v^r \tilde{g}(0) \frac{v^r}{r!} \tag{5.3}$$

so that the boundary conditions which must now be taken are

$$\begin{aligned} \partial_v^r \tilde{g}(0) &= r! \left[\psi_{n-r} \langle v^{n-r} k \rangle + \sum_{s=r}^n \phi_{sr} \langle v^{s-r} k \rangle \right] \\ &= r! \sum_{s=r}^n [\phi_{sr} + \psi_{n-r} \delta_{sn}] \langle v^{s-r} k \rangle \quad (r = 0, 1, \dots, n) \end{aligned} \tag{5.4}$$

generalizing (3.17). It is worthwhile to remark that, by virtue of the symmetry of $\phi(v, w)$, the nonlinear $(n + 1)$ th degree ordinary differential equation obtained for $\tilde{g}(v)$ coincides exactly with (3.8).

With respect to the two-folded character of the transition kernel, the additional terms contribute trivially to such structure. In fact

$$\tilde{K}(v, w) = \begin{cases} [\psi(v, w) + \phi(v, w)]k(v)k(w) & \text{for } w < v \\ [\psi(w, v) + \phi(v, w)]k(v)k(w) & \text{for } w > v. \end{cases} \tag{5.5}$$

However, this generalization extends the form of the polynomial factor in $\tilde{K}(v, w)$ to have terms with any power lower than or equal to n , i.e. allowing for a more complicated dependence in v and w for intermediate velocities.

5.2. Generalization II

This second generalization regards the possibility of introducing, as a new transition kernel, a linear combination of two kernels of the type considered in section 3. In fact, consider (1.1) with $K(v, w) = \alpha_1 K_1(v, w) + \alpha_2 K_2(v, w)$, taking

$$K_i(v, w) = \begin{cases} \psi_i(v, w)k_i(v)k_i(w) & \text{for } w < v \\ \psi_i(w, v)k_i(v)k_i(w) & \text{for } w > v \end{cases} \quad (i = 1, 2) \tag{5.6}$$

where the polynomials $\psi_i(v, w)$ satisfy (3.9) and (3.10) with respect to two integers n_i ($i = 1, 2$). The constants α_i are chosen so that $K(v, w)$ is always positive.

Equation (1.3) can now be written

$$f(v)[\alpha_1 k_1(v)g_1(v) + \alpha_2 k_2(v)g_2(v)] = S(v) \tag{5.7}$$

where two new dependent variables have been introduced, namely

$$g_i(v) = \int_0^v \psi_i(v, w)k_i(w)f(w) dw + \int_v^\infty \psi_i(w, v)k_i(w)f(w) dw \tag{5.8}$$

for $i = 1, 2$. Following the procedure detailed in section 3, both functions $g_i(v)$ can be made to satisfy

$$\partial_v^{n_i} g_i(v) = \gamma_i k_i(v) f(v) \tag{5.9}$$

with $\gamma_i = \partial_1^{n_i} \psi_i(v, v) - \partial_2^{n_i} \psi_i(v, v)$. Equation (5.7) then becomes

$$\alpha_1 \gamma_2 g_1(v) \partial_v^{n_1} g_1(v) + \alpha_2 \gamma_1 g_2(v) \partial_v^{n_2} g_2(v) = \gamma_1 \gamma_2 S(v) \tag{5.10}$$

which has to be solved with the compatibility condition

$$\gamma_2 k_2(v) \partial_v^{n_1} g_1(v) - \gamma_1 k_1(v) \partial_v^{n_2} g_2(v) = 0 \tag{5.11}$$

implied by (5.9), as from both g_1 and g_2 one should be able to derive the distribution function $f(v)$. Regarding the boundary conditions to be used to solve (5.10) and (5.11), they correspond exactly to those determined by (3.17), applied successively to $g_1(v)$ and $g_2(v)$.

The linear combination of two transition kernels like those studied previously, has led to a system of two differential equations. One of them is of the type already obtained in section 3; the other, instead, is a linear equation arising from a compatibility condition on the two new unknowns with respect to the sought distribution function. The extension to more complicated linear combinations follows straightforwardly.

The generalizations presented in this section are relevant for the application of the reduction method to specific problems in transport theory. In fact, the simple homogeneous-polynomial form given by (3.9) will not generally be appropriate for modelling real situations. More general forms, like (5.1) and (5.6), are, however, expected to fit real kernels with an acceptable degree of accuracy. In the next section, the problem of fitting the interaction models found in practice with the polynomial forms considered in this paper is worked out, paying particular attention to the asymptotic (large and small v) behaviour of the solutions to the integral equations.

6. Models for real interaction

It is evident that, when modelling a real interaction by means of a simplified form for the transition kernels, one desires to preserve in the distribution function satisfying the model equation some physically relevant features of the actual distribution. In particular, the behaviour of $f(v)$ for large v , i.e. the velocity dependence of the high energy tail, is a fundamental parameter in both the qualitative and quantitative analysis of a wide class of applied problems in kinetic theory [1]. Therefore, it is interesting to discuss which characteristics are required in the form of the transition kernel, in order to ensure the correct asymptotic behaviour in the distribution function. In the frame of the theory developed above, such conditions will fix the physically meaningful forms of the functions $k(v)$ and $\psi(v, w)$.

It can be seen that the asymptotic velocity dependence of $f(v)$, both for $v \rightarrow \infty$ and $v \rightarrow 0$, depends on the analogous features in the transition kernel (for fixed finite w) or, more precisely, on the microscopic probability transition $I(u)$. Indeed, expanding $K(v, w)$ in powers of w/v up to the first order gives

$$K(v, w) \sim \frac{1}{2vw} 2v^2 I(v) \frac{w}{v} = I(v) \tag{6.1}$$

which, replaced in (1.2), yields the behaviour of $f(v)$ for $v \rightarrow \infty$, i.e.

$$f(v) \sim \frac{S(v)}{I(v)\langle 1 \rangle}. \tag{6.2}$$

On the other hand, because of the symmetry of $K(v, w)$, its limiting value for $v \rightarrow 0$ is given by

$$K(v, w) \sim I(w) \tag{6.3}$$

so that, for small velocities, $f(v)$ is proportional to the source term, namely

$$f(v) \sim \frac{S(v)}{\langle I \rangle}. \tag{6.4}$$

Certainly, (6.2) and (6.4) display the asymptotic behaviour of the distribution function when the transition kernel is derived from a microscopic probability. However, when $K(v, w)$ is proposed as a mathematical model for the kinetic process, the features of $f(v)$ depend directly on the form of the transition kernel. Taking into account the form introduced in (3.1) and (3.9), and imposing the following physically reasonable limiting conditions

$$\lim_{v \rightarrow 0} v^r \int_0^v w^{n-r} k(w) f(w) dw = 0 \tag{6.5a}$$

$$\lim_{v \rightarrow \infty} v^r \int_v^\infty w^{n-r} k(w) f(w) dw = 0 \tag{6.5b}$$

for $r = 0, 1, \dots, n$, the asymptotic behaviour of $f(v)$ is

$$f(v) = \frac{S(v)}{k(v) \sum_{r=0}^n \psi_r v^r \langle v^{n-r} k \rangle} \quad \text{for } v \rightarrow \infty \tag{6.6}$$

and

$$f(v) = \frac{S(v)}{k(v) \sum_{r=0}^n \psi_{n-r} v^r \langle v^{n-r} k \rangle} \quad \text{for } v \rightarrow 0. \tag{6.7}$$

Now, if the model kernel $K(v, w)$ is supposed to imitate the situation obtained for a given probability transition $I(u)$, comparison between (6.2) and (6.6), and between (6.4) and (6.7), respectively, shows clearly that one should impose the conditions

$$k(v) \sum_{r=0}^n \psi_r v^r \langle v^{n-r} k \rangle \sim I(v) \quad \text{for } v \rightarrow \infty \tag{6.8a}$$

$$k(v) \sum_{r=0}^n \psi_{n-r} v^r \langle v^{n-r} k \rangle \sim \text{constant} \quad \text{for } v \rightarrow 0. \tag{6.8b}$$

Equations (6.8) are general conditions to be satisfied by the functions $k(v)$ and $\psi(v, w)$, in order to reproduce correctly the limiting behaviour of $f(v)$ for large and small velocities, respectively. Observe that only (6.8a) depends on the form of the microscopic probability transition to be imitated. For a power-law interaction potential $V(r)$, $I(u)$ behaves as u^α for $u \rightarrow \infty$, the constant α depending on the power of the radius r in the potential and the dimension of the space [6]. Therefore, since the polynomial in the LHS of (6.8a) exhibits also a power-law behaviour, $k(v)$ can be taken to have the form

$$k(v) \sim v^p \quad \text{for } v \rightarrow \infty \tag{6.9}$$

where $p = \alpha - R$, and R is the maximum value of the index r so that $\psi_r \neq 0$. In fact, the corresponding term in the polynomial $\psi(v, w)$ dominates its v -dependence for high velocities.

On the other hand, (6.8b) fixes the form of $k(v)$ for vanishing velocities to be also a power law, i.e.

$$k(v) \sim v^q \quad \text{for } v \rightarrow 0 \tag{6.10}$$

now, with $q = R - n$, as the velocity dependence of $I(u)$ is not involved in (6.8b). Equations (6.9) and (6.10) are general criteria for choosing the form of $k(v)$ and the values of n and R . Note, however, that these conditions do not determine uniquely such values, so that there remains a wide variety of functions from which one can select a model for the physical problem.

Take, as an example, $I(u) = u^\alpha \equiv u$, i.e. the 'hard-sphere' interaction. As said in section 4, the kernel generated by this probability transition from (1.3), namely, (4.11), has the form required to apply the theory developed in section 3. For this case, in fact, one finds that $k(v) = v^{-1}$, so that $p = q = -1$, in agreement with the values $\alpha = 1$, $n = 3$ and $R = 2$. But suppose that one is interested in proposing a new model, imitating the 'hard-sphere' interaction for large and small velocities. Particular values satisfying the conditions imposed through (6.9) and (6.10) for $\alpha = 1$ are $p = q = 0$, $n = 1$ and $R = 1$. Indeed, taking $k(v) = 1$ and $\psi(v, w) = v$, one obtains

$$K(v, w) = \begin{cases} v & \text{for } w < v \\ w & \text{for } w > v \end{cases} \tag{6.11}$$

or, more briefly, $K(v, w) = \max(v, w)$. This is the so-called 'maximum' model, which displays a particularly simple form, in view of the values selected for the relevant velocity powers in the functions $k(v)$ and $\psi(v, w)$. This kernel cannot be derived from a particular form of $I(u)$. Its validity as an imitation of the 'hard-sphere' interaction is analysed in detail in [5].

7. Conclusion

In the frame of the extended kinetic problem considered in the paper, the method presented here introduces a tool for reducing a nonlinear integral equation to a nonlinear ordinary differential equation. The convenience of such reduction resides in the fact that a well developed toolkit of analytical and numerical methods does exist for differential equations, whereas integral problems are usually treated approximately [1]. The only restriction in the form of the kinetic equation regards the explicit expression of the model transition kernel, which preserves however the two-folded structure present in real interaction kernels. Nevertheless, one should become convinced that such a restriction is not particularly severe. In fact, the differential problem is independent of the factored function $k(v)$, which can be fixed to reproduce properly the asymptotic behaviour of the transition kernel, as required for obtaining a realistic description of the physical problem under study. Furthermore, the almost free choice in the degree of the polynomial $\psi(v, w)$ multiplying the kernel, as well as the wide freedom when selecting its coefficients, permits a great variety of fine detail for $K(v, w)$ at intermediate velocities. A more complicated form for $k(v)$, as well as the discussed generalizations, can also contribute to these details. In fact, as seen in the last section, even when limiting conditions are imposed in order to properly represent real interactions, the remaining freedom in the choice of the transition kernel is indeed wide. In this sense, the reduction method can be considered to apply for a relevant class of transition kernels.

Finally, one must bear in mind that the theory developed in the paper involves essentially the integral operator in the kinetic equation. This fact suggests the extension of the reduction method to a great variety of kinetic problems, in particular, not necessarily stationary ones. For instance, the evolution equation associated with the source-removal problem described in the introduction, namely

$$\partial_t f(v, t) = S(v, t) - f(v, t) \int_0^\infty K(v, w) f(w, t) dw$$

can be also reduced by the same procedure, obtaining now a partial differential equation. The method also applies to some generalizations of (1.2), for example

$$r(v)f(v) + f(v) \int_0^\infty K(v, w)f(w) dw = S(v)$$

where, in the considered extended kinetic problem, the first term represents removal interaction with a host medium through which the gas under study diffuses. This last equation, which reduces also to an ordinary differential form, is related with the Hammerstein-Chandrasekhar problem [2, 9].

Furthermore, some linear integral problems could also be treated applying the same idea, i.e. the association of the integral operator with a new dependent variable. This is the case, for instance, in the Wiener-Hopf problem, appearing in neutron transport theory [1].

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